Very often, for the distribution of Z the Gamma distribution is used.

This idea can be generalized in the bivariate case. If the two individuals share the same frailty Z we obtain the Shared Frailty model. If the frailty values are different but dependent, we obtain Correlated Frailty model.

3. THE SHARED FRAILTY MODEL

In this model we assume that both individuals in a pair share the same frailty Z. The two lifespans are assumed to be conditionally independent with respect to the shared frailty Z. According to Wienke (2001), the bivariate survival function can be derived as follows:

$$S(x_1, x_2 \mid Z) = S_1(x_1)^Z S_2(x_2)^Z = e^{-ZH_1(x_1)} e^{-ZH_2(x_2)} = e^{-ZH_1(x_1) - ZH_2(x_2)},$$

where $H_i = \int_0^x \mu_0(s)ds$ is the cumulative baseline hazard function. Averaging with respect to Z we have

$$S(x_1, x_2) = ES(x_1, x_2 \mid Z) = L(H_1(x_1) + H_2(x_2)),$$

where L is the Laplace transformation of Z. It is a standard assumption that the frailty distribution is the Gamma distribution with mean 1 and variance σ^2. In this case for the bivariate survival function we have

$$S(x_1, x_2) = (S_1(x_1)^{-\sigma^2} + S_2(x_2)^{-\sigma^2} - 1)^{-\frac{1}{\sigma^2}}.$$

This is just a different derivation of the result of Yashin and Iachine (1999) mentioned above.

4. THE CORRELATED FRAILTY MODEL

Let us suppose that the mortality among the individuals in the population depends on an unknown random variable Z. As we mentioned above the hazard function can be expressed as $\mu(x \mid Z) = Z\mu_0(x)$. It is rational to suppose that Z is Gamma distributed with $EZ = 1$. On the one hand, this allows us to have an expression for the survival function, on the other hand, by changing the parameters of the Gamma distribution a large class of probability distributions can be dealt with.

To represent the dependence between the related individuals, let us suppose that the frailty can be decomposed as $Z = Z_g + Z_e$. Here Z_g represents the genetic influence and Z_e represents the environment influence on the lifespan of the individual. As Z is supposed to be Gamma distributed, we can assume that Z_g and Z_e are also Gamma distributed. So, for related individuals with the same genetic information and different environment we have that the correlation between the two frailty parameters is due to the common variable Z_g. In opposite, if the individuals are with different genetic information, but living in the same environment, the corresponding variable Z_e will be the same.